RECTANGULAR HYPERBOLA - CIRCLE
GEOMETRIC PROPERTIES AND FORMAL ANALOGIES
- Part XXIII -

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Abstract

We review some selected elementary properties of the rectangular hyperbola and compare to those of the circle. The rectangular hyperbola has great analogy and many corresponding characteristics. An important application is the analogic definition of hyperbolic trigonometry on the model of circular trigonometry and circular functions.

1 Standard Equation of the circle and rectangular hyperbola

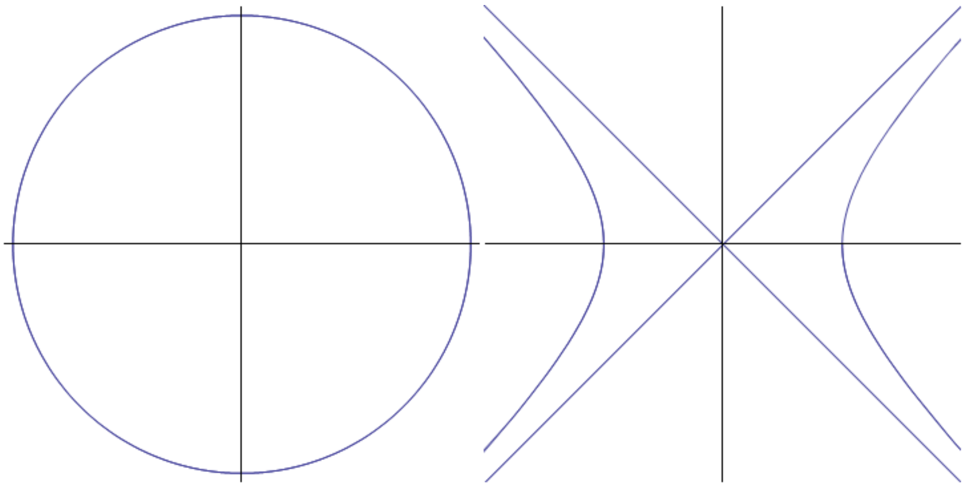

Figure 1: The Circle and the Rectangular Hyperbola

The equation of these curves in orthogonal coordinates are :

$$\boxed{x^2 + y^2 = a^2 \qquad \text{Circle}}.$$

$$\boxed{x^2 - y^2 = a^2 \quad \text{or} \quad x.y = a^2/2 \qquad \rightarrow \text{Rectangular Hyperbola}}.$$

2 Integration of x^n for n=-1

In the seventteenth century Cavalieri gave the general well known formula for the integration of x^n between 0 and a equivalent to :

$$\int_0^a x^n dx = \frac{a^{(n+1)}}{n+1}$$

So the integral of a power of is also a power. But for n=-1 this integration is not valid, the denominator is zero. This case led geometers to exponential and hyperbolic functions. Gregory of St Vincent in his work of 1647 'Opus Geometricum' had showed a property of the areas under the rectangular hyperbola $y = 1/x$ between two parallel lines to y axis and the asymptote Ox. This is not a trivial result and in reference (2) the proof is reconstructed in 9 steps since Gregory of St Vincent just gives geometric proofs of relation between areas under the hyperbola. The path to the global understanding needed to wait until Euler's works. RP Burn in (4) explains that six steps are necessary to define logarithm by means of the rectangular hyperbola. It is necessary to adopt log 1=0, hyperbola must have equation $y = 1/x$ and the logarithm must be defined in the continuum.

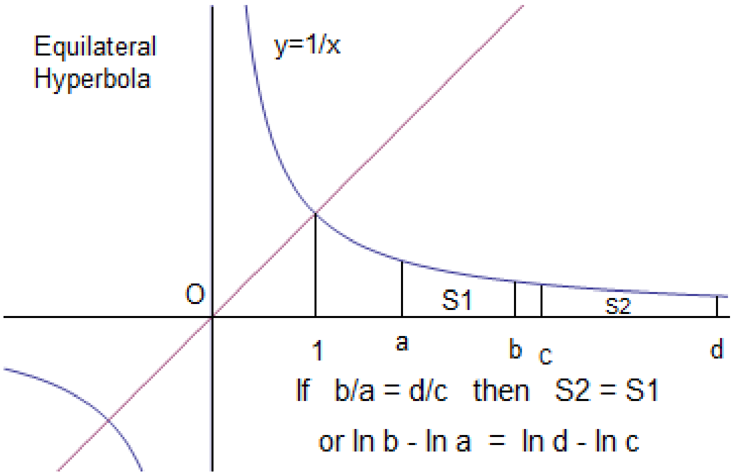

Figure 2: Property of Gregory of St Vincent

$$\text{if } \frac{b}{a} = \frac{d}{c} \text{ then Area S1(a,b) = Area S2(c,d).}$$

This theorem can be written in modern notation as :

$$\int_a^b \frac{dx}{x} = \int_{c.a}^{c.b} \frac{dx}{x}$$

And Felix Klein proposed to use this relation to define the natural logarithm. We can say in a succinct formula tha the areas under the hyperbola behave like a logarithm. But the notion is not really easy to explain for the first time to students.

3 Some equations of the rectangular hyperbola

The rectangular hyperbola is a conic with excentricity $e = \sqrt{2}$. In an orthonormal system we have the normal well known equation :

$$x^2 - y^2 = 1$$

In the system composed of the two orthonormal asymptotes the equation is :

$$X.Y = \frac{1}{2}$$

The rectangular hyperbola is also a sinusoidal spiral $\rho^n = a^n. \sin(n\theta)$ for $n = -2$. so we have in polar coordinates :

$$\rho^{-2} = a^{-2}.sin(2\theta)$$

or

$$\rho^2 = \frac{a}{\sin(2\theta)}$$

which we can write :

$$\rho^2.\sin(2.\theta) = 2.a^2 = 2.\rho.\cos\theta.\rho.\sin\theta = 2.X.Y$$

So we have $X.Y = a^2$ with $a = \frac{1}{\sqrt{2}}$ to fall back on the above equation. Another way to find this equation is to use the expression of the distance to a line in the plane :

$$d = \frac{|ux + vy + c|}{\sqrt{u^2 + v^2}}$$

The two asymptotic lines of the rectangular hyperbola are $x + y = 0$ and $x - y = 0$ so: $X = \frac{x+y}{\sqrt{2}}$ and $Y = \frac{x-y}{\sqrt{2}}$ and we get :

$$X.Y = \frac{x^2 - y^2}{2} = a^2$$

4 Central sector Area and area under the rectangular hyperbola $y = 1/x$

There is an important relation between area of the central sector limited by an arc of hyperbola and area inside two ordinates, x-axis and the arc. For any point M on the first branch of the hyperbola $y = 1/x$ the triangle OHM (H abcissa of M) has a constant area =1/2 since $x.y = 1$. If we remove successively one of the two triangles (of same area OAI and OBJ) of the area OIJBO the remaining parts have same area so :

Central sector Area [OIJ] = under arc Area [AIJBA]

This last area is equal if OA=1 to ln x = $\int_1^x \frac{dt}{t}$, so $\log 1 = 0$. If we denote this area as $u = \log x$ or $x = e^u$ and use u as the new parameter. Then the parametric equations for the rectangular hyperbola $x.y = 1$ are $x = e^u$ and $y = e^{-u}$. We verify that area A under the curve between 1 and x is u since

$$A = \int_1^x ydx = \int_0^u e^{-h}e^h dh = \int_0^u 1dh = u = \log x$$

We search for the equations of this rectangular hyperbola in cartesian frame of 1st and 3dr bisectors. And get :

$$X = \frac{x + y}{\sqrt{2}} = \frac{e^u + e^{-u}}{\sqrt{2}} = \sqrt{2}.\cosh u$$

$$Y = \frac{x - y}{\sqrt{2}} = \frac{e^u - e^{-u}}{\sqrt{2}} = \sqrt{2}.\sinh u$$

It is the standard parametrization up to a scale factor : $\sqrt{2}$.
Using polar equation of the rectangular hyperbola $\rho^2 \sin 2\theta = 1$ to compute the central sector area, we find :

$$A = \frac{1}{2}\int_\theta^{\pi/4} \rho^2 d\theta = \frac{1}{2}\int_\theta^{\pi/4} \frac{d\theta}{\sin 2\theta} = \frac{-1}{4}\log\tan\theta = u/2$$

The caustic by relection of the rectangular hyperbola y=1/x for the light rays parallel to Oy axis is the bipartite curve (we use parameter t=x so x=t, y=1/t):

$$x = \frac{3}{2}.t \qquad y = \frac{3 + t^4}{4t}$$

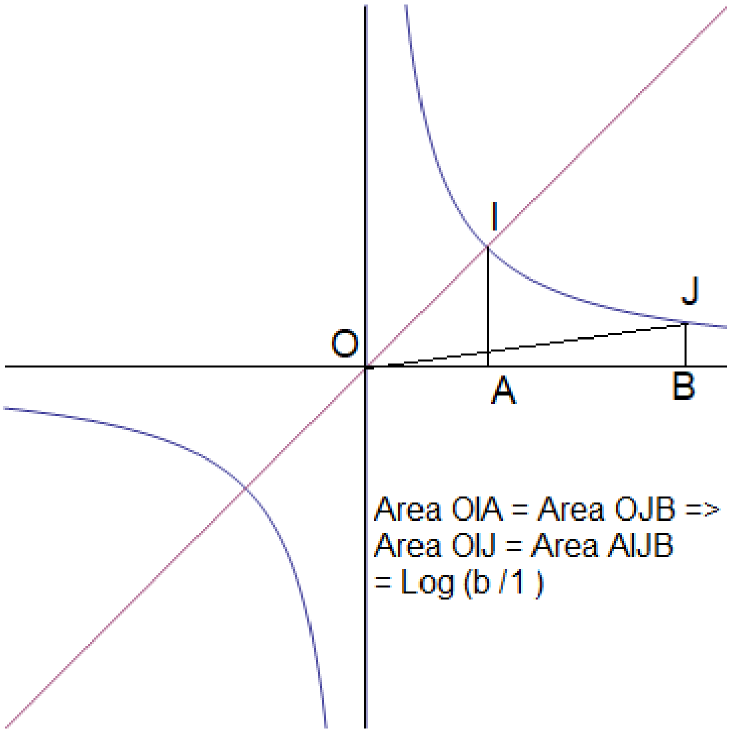

Figure 3: Central sector and area under the rectangular hyperbola arc IJ

The wheel corresponding to the rectangular hyperbola $y = 1/x$ the pole running along the asymptote x'x is the spiral given if t=x by the parametric equations :

$$\rho = \frac{1}{\sqrt{2t}} \qquad\qquad \theta = t$$

5 Circular and hyperbolic functions

In circular trigonometry the function sinus, cosinus and tangent are defined as ratios of length read on the trigonometric circle of radius one. They are maps between angles and length

In hyperbolic trigonometry the function sinus H, cosinus H and tangent H are defined as ratios of length read on the rectangular hyperbola with distance between the center and vertex equal to one.

We have some important formulas for sinus and cosinus coming from complex numbers in the plane ($\imath^2 = -1$) and analysis :

$$e^{\imath.\theta} = \sin\theta + \imath\sin\theta \qquad ; \qquad e^{-\imath.\theta} = \sin\theta - \imath\sin\theta$$

$$\sin\theta = \frac{e^{\imath\theta} - e^{-\imath\theta}}{2.\imath} \text{ and } \cos\theta = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{\imath\theta} - e^{-\imath\theta}}{\imath.(e^{\imath\theta} + e^{-\imath\theta})} = \frac{1}{\imath}.\tanh\imath.\theta$$

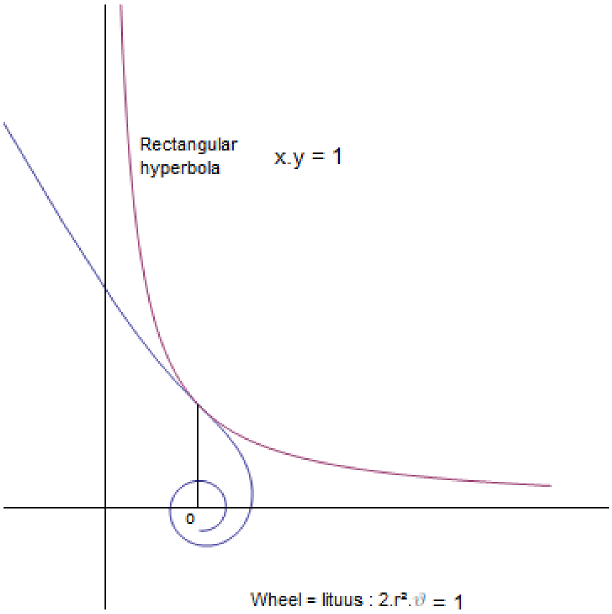

Figure 4: Wheel for rectangular Hyperbola y=1/x (Lituus $2r^2.\theta = 1$).

The corresponding formulas for hyperbolic functions are :

$$\sinh t = \frac{e^t - e^{-t}}{2} \text{ and } \cosh t = \frac{e^t + e^{-t}}{2}$$

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

Since we have :

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad \cosh^2 t - \sinh^2 t = 1$$

The parametrization of the two curves are :
For a point M on the circle :

$$\boxed{M(\theta) \leftrightarrow (\cos\theta, \sin\theta)}.$$

For a point P on the rectangular hyperbola :

$$\boxed{P(t) \leftrightarrow (\cosh t, \sinh t)}$$

Other formulas linking the two species of function are ($\imath^2 = -1$) :

$$\boxed{\sin \imath x = \imath.\sinh x \quad , \ \cos \imath x = \cosh x \quad \text{and} \ \tan \imath x = \imath.\tanh x}$$

$$\boxed{\sinh \imath x = \imath.\sin x \quad , \ \cosh \imath x = \cos x \quad \text{and} \ \tanh \imath x = \imath.\tan x}$$

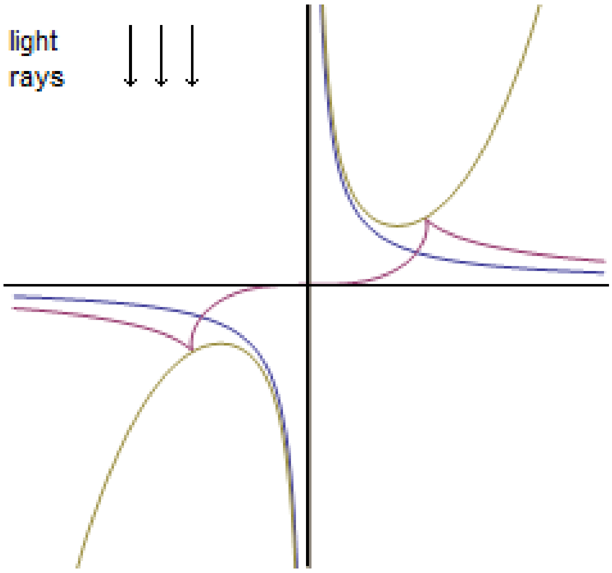

Figure 5: Caustic by reflection light rays parallel to y-axis

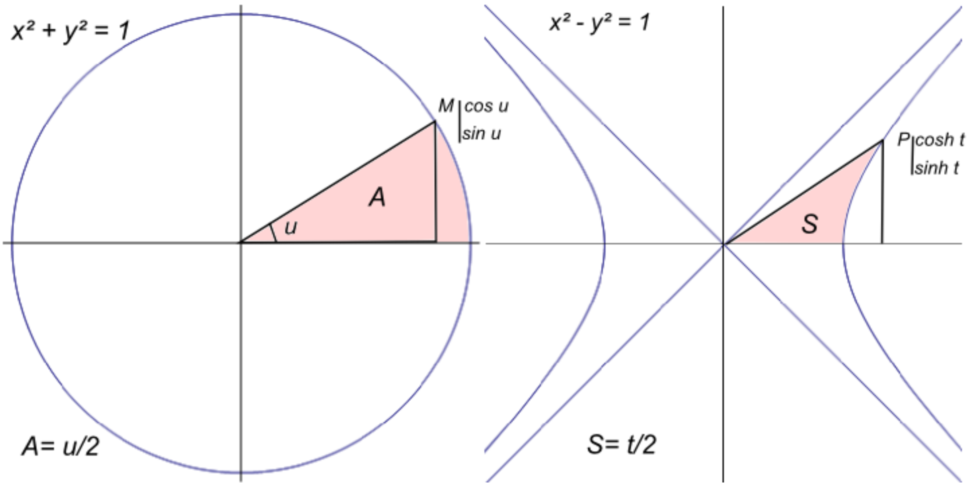

Figure 6: Circular and hyperbolic functions

6 Some geometric remarks

If we replace y by i.y in the circle circle equation we get the hyperbola equation : it is an imaginary affinity or an imaginay rotation on the y-axis that trasform the circle in the rectangular hyperbola.
In the complex plane the unit circle is given by the equation $z = e^{\imath.\theta}$ and the similar equation of the rectangular hyperbola is :

$$z = \sqrt{\cosh(2t)}.e^{\imath.\arctan\tanh t}$$

In paper (8) of 1846 Abel Transon mentions some analogies between the circle and the rectanguar hyperbola :

- "The tangent to the circle is orthogonal to line from the center to the current point. The tangent to the restangular hyperbola is anti parallel to the line joining the center to the current point."

- "The distance from center of the circle or rectangular hyperbola to a point of a curve and multiplied by the distance from centre to the tangent at the same point is constant."

6.1 Center of curvature at a point of rectangular hyperbola and on the circle

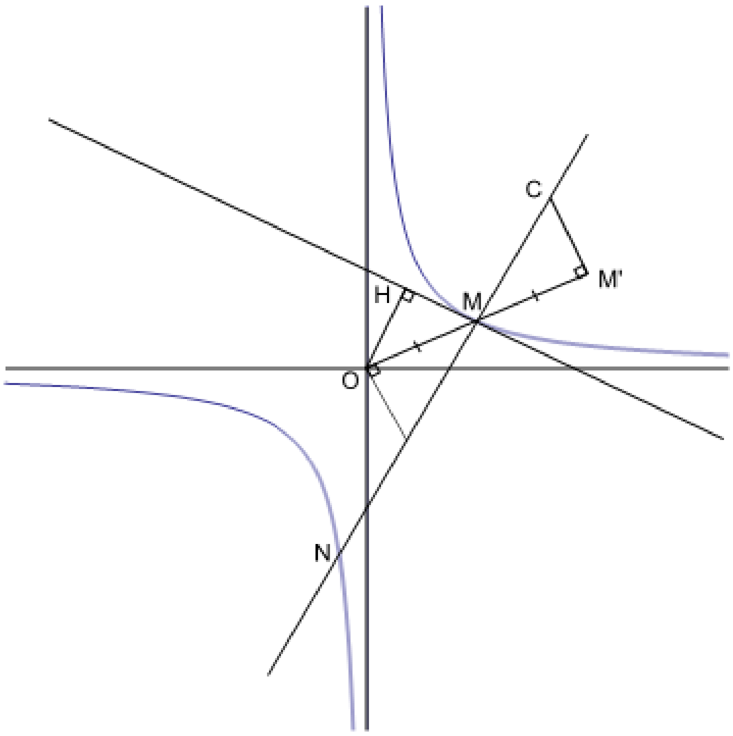

Figure 7: Radius of curvature of rectangular hyperbola : x.y=1

A trivial property of the circle of center O relative to the radius of curvature is the following. A diameter passing through M on a circle, we call N the second point diametrically opposite on the circle. We have a simple relation (radius equal to half diameter):

$$\overline{MN} = 2.\overline{MO}$$

Now if a rectangular hyperbola is given by equation :

$$x.y = a^2$$

Then some computations give the equation of the normal at current point M :

$$(X - x) + (Y - a^2/x).(-a^2/x^2) = 0$$

The two points of intersection of this normal are M and N. We know $M(x, y = a^2/x)$ and we look for $N(X, Y = a^2/X)$ so we solve equation in X :

$$x^3.X^2 + (a^4 - x^4)X - a^4x = 0$$

x is a solution (correponds to M) the product of roots is $-a^4/x^2$ so the second root is $-a^4/x^3$. Point N has coordinates $(-a^4/x^3, -x^3/a^2)$.
The unit vector of the normal is :

$$\overrightarrow{N} \quad (a^2/\sqrt{a^4+x^4} = \sin\phi, \quad x^2/\sqrt{a^4+x^4}) = \cos\phi$$

The radius of curvature of the hyperbola at M is :

$$R_c = MC = \frac{(x^4+a^4)^{3/2}}{2.a^2.x^3}$$

The vector MC is :

$$\overrightarrow{MC} \quad (\frac{a^4+x^4}{2.x^3} = R_c\cos\phi, \quad \frac{a^4+x^4}{2.a^2.x} = R_c\sin\phi)$$

The vector NM is :

$$\overrightarrow{NM} \quad (\frac{a^4+x^4}{x^3}, \quad \frac{a^4+x^4}{a^2.x})$$

So if N is the second point of intersection of the normal at M on a rectangular hyperbola, the radius of curvature at M verifies the relation :

$$\overline{MN} = -2.\overline{MC}$$

This formula is similar to the one above for the circle.

7 The gudermanian : the real link between the two worlds

The Gudermanian function, which links real circular and real hyperbolic functions, gives an equivalence between trigonometric functions and hyperbolic functions of parameter t, the hyperbolic argument.

7.1 Some properties of gudermanian functions

The gudermanian is defined by the following equations :

$$\boxed{\tanh(t/2) = \tan(u/2)}$$

On the graph of Gudermanian the line AMP is the geometric correspondance between the values of $0 < u < \pi/4$ and the quarter of rectangular hyperbola in the first quadrant. It gives directly the above formula. This formula is equivalent to each of these :

$$\boxed{\sinh t = \tan u} \ \boxed{\cosh t = 1/\cos u} \ \boxed{\tanh t = \sin u}$$

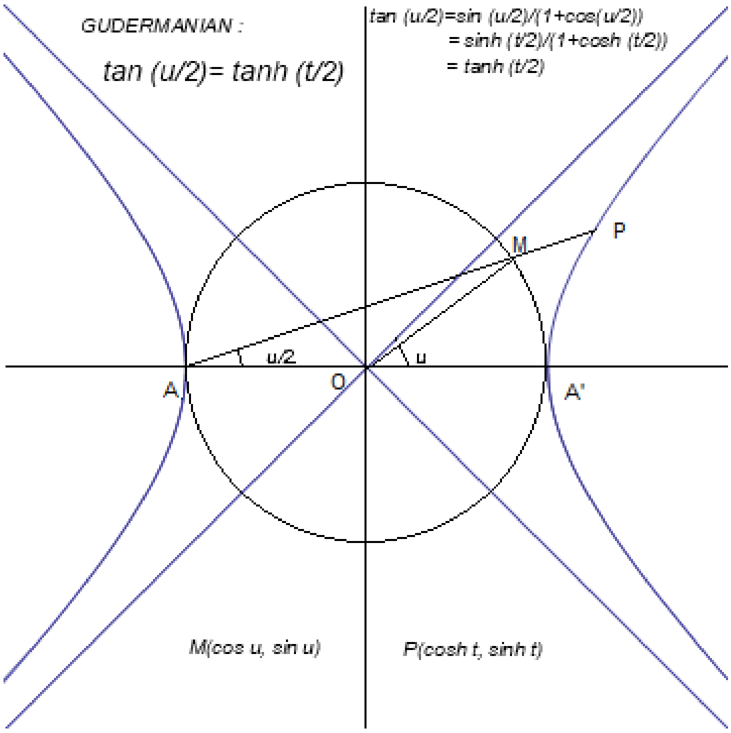

Figure 8: Gudermanian : Circular↔ hyperbolic functions

The direct gudermanian is :

$$u = Gd(t) = \arctan(\sinh t)$$

and the inverse gudermanian :

$$t = Gd^{-1}(u) = \arg\sinh(\tan u)$$

equivalent to the above formulas. The function Gd(t) is defined in $]-\infty, +\infty[$ and is in bijection with $]-\pi/2, +\pi/2[$. Gd^{-1} is defined on $]-\pi/2, +\pi/2[$ with the same precautions as for inverse trigonometric functions.

Other formulas are :

$$Gd(t) = \int_0^t \frac{dv}{\cosh v} \qquad \text{and} \qquad Gd^{-1}(u) = \int_0^u \frac{dv}{\cos v}$$

8 Rectangular triangle and pseudo-rectangular triangle

A rectangular triangle can be inscribed in a half circle, the hypothenuse lies on the horizontal diameter, the summit of right angle is on the circle. A classical property property gives HM^2= -HA'.HA so y^2=-(x+1) (x-1) and $x^2 + y^2 = 1$.

For the rectangular hyperbola in standard position there is a similar property (see fig.9). Consider a circle passing through A' an A and the right

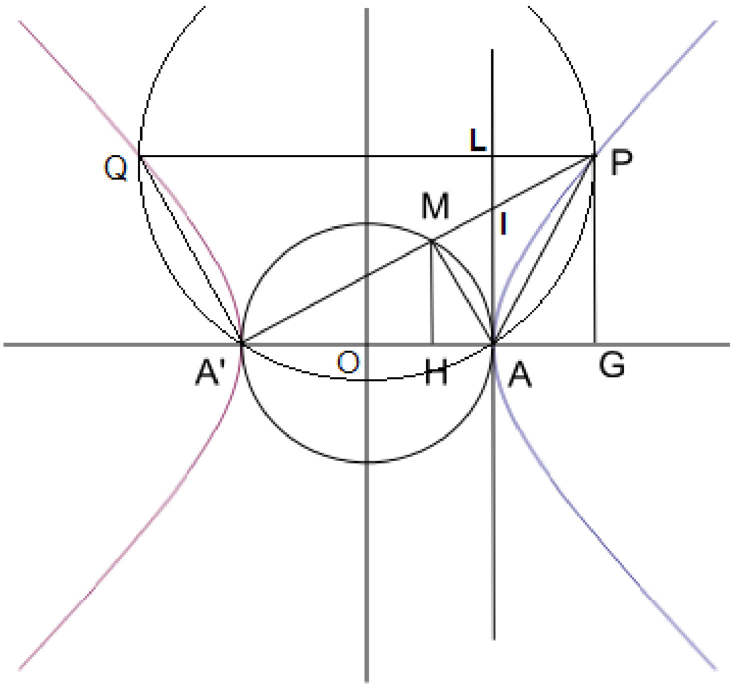

Figure 9: Rectangular and pseudo-rectangular triangles

tangent parallel to y-axis. On the diameter parallel to the x-axis call P the contact with vertical tangent GP. The power of G is GA'.GA=GP^2 so (x+1).(x-1)=y^2 or $x^2 - y^2 = 1$. The locus of P is a rectangular hyperbola. The retangular triangles QA'P or QAP inscribed in the rectangular hyperbola are sometimes called pseudo-rectangular by analogy with the rectangular triangle inscribed in a circle.

9 Homologies exchanging the circle $x^2 + y^2 = a^2$ and rectangular hyperbola $x^2 - y^2 = a^2$.

A projective homology with center A' and axis of fixed points, the parallel line to Oy and exchanges the circle and the rectangular hyperbola. A line through A' cuts the circle at M the homology axis at I and the rectangular hyperbola at P. The cross-ratio (A', I; M, P) = -1 so it is a harmonic homology and the angles MAI and IAP are equal. The symetrical homology w.r.t. axis-Oy with center at A also exchanges the two curves.
We know, by parallel projection, that cross-ratio (A', A; H, G) = -1 so $OA^2 = OA'^2 = \overline{OH}.\overline{OG} = 1$ and so $x_{rh} = OG = 1/\cos\theta$.
In the triangle APG the angle $\widehat{APG} = \theta/2$ and so $AG = 1/\cos\theta - 1$ and $AG/GP = \tan\theta/2$. Some trigonometric computations give finally $PG = y_{rh} = \tan\theta$. On the fig.9 we note that O, M and L are on a line.
So another parametrizaton of the rectangular hyperbola with circular func-

tions is :

$$\boxed{x_{rh} = 1/\cos\theta \qquad y_{rh} = \tan\theta}.$$

These are just the consequence of gudermanian relations and since

$$1/\cos^2\theta = 1 + \tan^2\theta$$

we have $x^2 - y^2 = 1$ which is the equation of the rectangular hyperbola. It must noted that the angle $\theta = \widehat{AOM}$ so $\widehat{AA'M} = \theta/2$ and we have :

$$\frac{GP}{A'G} = \frac{\sinh t}{1+\cosh t} = \tanh t/2 \qquad \frac{MH}{A'H} = \frac{\sin\theta}{1+\cos\theta} = \tanh\theta/2$$

So the two parameters θ and t are linked by the Gudermanian relations (see above).

10 Biangular and bipolar coordinates

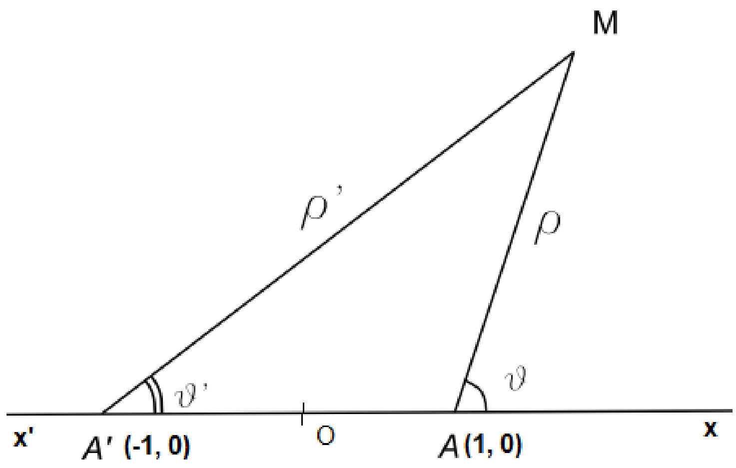

Figure 10: Bipolar (ρ, ρ') and biangular (θ, θ') coordinates with poles A , A'

Biangular coordinates use two fixed points A(-1, 0) and A'(1, 0) and two lines rotating around these points. The two angles are measured from the direction of defined by the two points (x-axis). When the lines intersect the common point is M. When they are parallele the point is at infinity : it is an asymptotic direction.
The coordinates are two angles $\theta = \angle$ MAx and $\theta' = \angle$MA'x . Line x'Ox is the horizontal axis. When $\theta = \theta' + k\pi$ we get the asymptotic directions of the curve. The biangular equation of a curve is a relation between these angles f(θ, θ').
Bipolar coordinates have important connections with the biangular ones. The bipolar coordinates of M, with same poles A and A', are two length

: ρ=AM and ρ'=A'M. Here the condition for the existence of point M is the triangular inequality: $\rho + \rho' > AA' = 2$.
Given a biangular or bipolar equation of a plane curve, it is sometimes easy to find orthogonal trajectories - see (1). If a family of cuves in bipolar is : $f(\rho, \rho') = h$, with h a constant, then :

$$\frac{\partial f}{\partial \rho}.d\rho + \frac{\partial f}{\partial \rho'}.d\rho' = 0$$

to find the orthogonal trajectories of this family of curves we change in this equation the ratio $d\rho/d\rho'$ by the ratio $\rho.d\theta/\rho'.d\theta'$. So for orthogonal trajectories we have :

$$\frac{\partial f}{\partial \rho}.\rho.d\theta + \frac{\partial f}{\partial .\rho'}\rho'.d\theta' = 0$$

In the triangle MA'A we have :

$$\frac{\rho}{\rho'} = \frac{\sin\theta}{\sin\theta'}$$

We can eliminate the ratio ρ/ρ' between the above equations and the resulting differential equation is the one of orthogonal trajectories in biangular coordinates this time.
For the general Cassinian ovals $f = \rho^n.\rho'^p = h$ then

$$n.\rho^{n-1}\rho'^p.d\rho + p.\rho^n.\rho'^{p-1}.d\rho' = 0$$
$$n.\rho^n\rho'^p.d\theta + p.\rho^n\rho'^p.d\theta' = 0$$
$$n.d\theta + p.d\theta' = 0$$

and so :

$$n.\theta + p.\theta' = \text{K} = \text{constant}$$

11 Pencils of circles in biangular and bipolar coordinates

The current point M on any circle of the Poncelet pencils has a constant $\angle$ A'MA $= \alpha$ (or $\pi - \alpha$ on the other circular arc). That's a well known property of the circle (the inscribed angle theorem = half of center angle). Another way of considering this fact in angular coordinate, with θ and θ' as shown on the figure above, is to note that angular equation :

$$\theta - \theta' = \alpha \text{ constant}$$

This is a special case of general Cassinian ovals for n=+1, p=-1 so the orthogonal trajectories are defined by $\rho^{+1}.\rho^{-1} = \frac{\rho}{\rho'} = k$ and correspond to the well known orthogonality of pencils of Poncelet/Apollonian circles for the same base or limit points.
An analog statement is valid for the conjugate Apollonian pencil of circles. The bipolar equation is :

$$\frac{\rho}{\rho'} = \text{k (constant)}$$

For Cassinian ovals (n=p=1) the biangular equation of $\perp$-trajectories :

$$\theta + \theta' = \alpha$$

as we will see further that these curves are rectangular hyperbolas passing at A and A'.
Using a complex representation of the polar-vectors AM=(ρ, θ) and A'M(ρ',ϕ) which can be in trigonometric complex form :

$$z = \rho.e^{i\theta} \qquad z' = \rho'.e^{i\theta'}$$

The complex equation of the pencil is $f = \rho^n.\rho'^p = h$ and:

$$z^n.z'^p = \rho^{n.\theta}.\rho'^{p.\theta'}.e^{\imath.(n.\theta+p.\theta')} = r.e^{\imath.\sigma}$$

In the complex plane $x + \imath y = r.e^{\imath.\sigma}$ the curves r= constant are circles centered at origin and σ=constant lines through origin. Since these are two families of $\perp$-trajectories, the curves generated by complex function $F = z^n.z'^p = r.e^{\imath\sigma}$ are curves defined by condition $\sigma = n.\theta + p.\theta' =$ constant.

12 pencils of circles of Poncelet and Apollonius

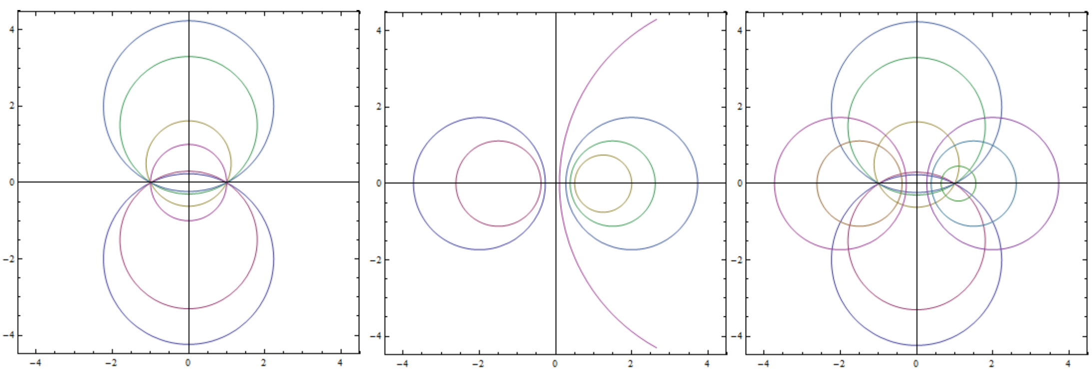

Figure 11: Circle pencils of Poncelet and Apollonius

A Poncelet pencil of circles in the plane is the set of circles passing through two fixed points A' (-1, 0) and A (+1, 0). All centers lay on mediator of AA' the y-axis. The equation of this pencil is :

$$x^2 + y^2 - 2\lambda y - 1 = 0$$

All these Poncelet circles have an orthogonal family of curves called Apollonian pencil. The two pencils are called conjugate. Instead of fixed points this pencils has two limit points A and A' and the centers are on the x-axis, the fixed points of Poncelet pencil. This pencil, as we have see above, can be defined in biangular coordinates by $\theta - \theta' = \alpha =$ constant. The relation between λ and θ is $\lambda = 1/\tan\alpha$. The parameter λ is the y of the center of a circle of the Poncelet pencil.
The equation of this Apollonian family of circles is :

$$x^2 + y^2 - 2\mu x + 1 = 0$$

These circles have no common point and the y-axis is the radical axis of any couple of circles choosen in the Apollonian pencil. The parameter μ is the x of the center of an Apollonian circle of the pencil.
This pencil of apollonian circles is characterized by the constant ratio MA/MA'=k where A'(-1, 0) and A(+1, 0) are the limit points of the pencil.

13 pencils of rectangular hyperbola and orthogonal pencil of Cassini ovals

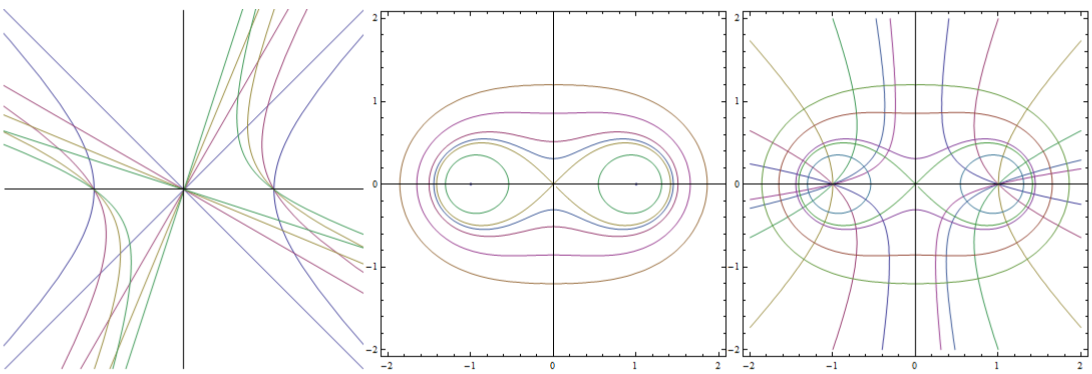

Figure 12: Rectangular Hyperbola Poncelet pencils and Cassini ovals

A. Caylay in a paper of 1862 (9) explains that "every conic which passes through the points of intersection of two rectangular hyperbolas is a rectangular hyperbola". If a pencil of conics has two rectangular hyperbolas for base conics (U(x, y)=0 and V(x, y)=0) then all the curves of the pencil are rectangular hyperbolas. If $U = x^2 - y^2 - 1 = 0$ and $V = x.y = 0$ ($x.y = 0$ is the limit case of two orthogonal lines) then :

$$x^2 - y^2 - 2\mu.x.y - 1 = 0$$

Is the equation of a pencil of rectangular hyperbolas passing through two real points A' (-1, 0) and A (+1, 0).
This pencil of hyperbolas is equivalent to the set of curves defined by the biangular equation :

$$\theta + \theta' = C \qquad \text{C constant}$$

As we have seen above the corresponding "pencil of Apollonius" for rectangular hyperbolas is the pencil of Cassini ovals with foci or poles in A' and A. The bipolar equation is :

$$\rho.\rho' = \text{ k (constant)}$$

This is bipolar equation of Cassini ovals (see fig.12).

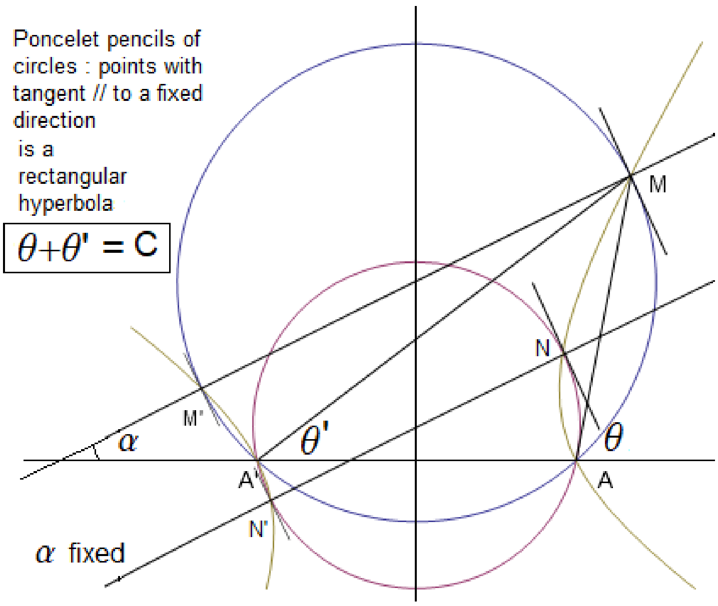

Figure 13: Rectangular Hyperbola and Poncelet pencils of circles

14 Rectangular hyperbolas related to pencils of circles

Poncelet an Apollonian pencils of circles have a close relation with the rectangular hyperbolas.
Theorem 14 of ref. (7) : "Given a pencil of circles of intersecting 'Poncelet' and non-intersecting 'Apollonian' type and a direction the diametral

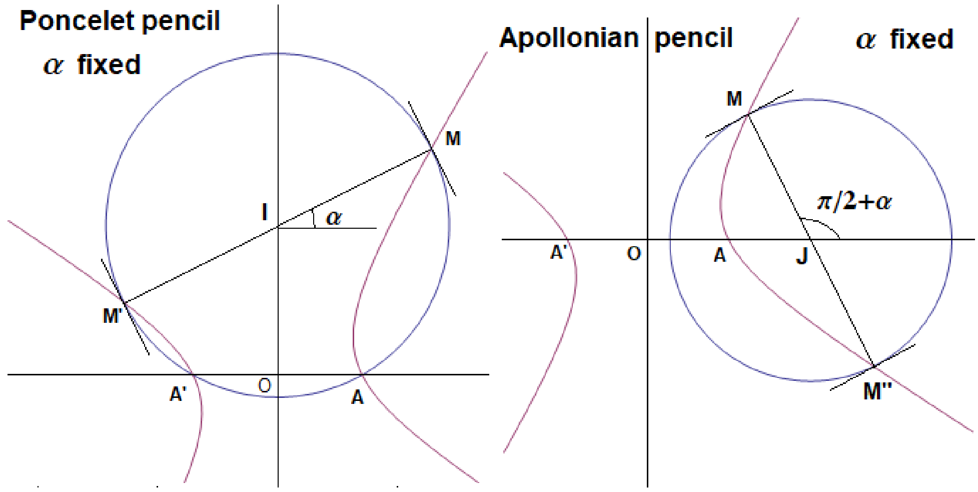

Figure 14: Pencils of circles and rectangular hyperbolas

points of the diameters of members-circles, which are parallel to this direction generate a rectangular hyperbola".

14.1 The case of Poncelet pencil

We choose a circle passing through A' and A with center at I $(0, \lambda)$ and M (α) is the point of tangency with fixed direction. The coordinates of current pont M on the poncelet circle of radius $R = \sqrt{\lambda^2 + 1}$ are :

$$x = \sqrt{\lambda^2 + 1}.\cos\alpha \qquad y = \sqrt{\lambda^2 + 1}.\sin\alpha$$

If we eliminate λ we find :

$$\lambda^2 = \frac{x^2}{\cos^2\alpha} - 1 \qquad y = \lambda + x\tan\alpha$$

$$(y + x\tan\alpha)^2 = (x^2 - \cos^2\alpha)(1 + \tan^2\alpha)$$

$$x^2 - y^2 - 2\tan\alpha.x.y - 1 = 0$$

This is the equation of a pencil of rectangular hyperbolas with $\lambda = \tan\alpha$. For $\alpha = 0$ then $x^2 - y^2 - 1 = 0$ the standard, and for $\alpha = \pi/2$ the limit case of a couple of orthogonal lines.

14.2 The case of Apollonian pencil

A circle of radius R in the apollonian pencil of circles with limit points A' (-1, 0) and A(+1, 0) and center at $(\mu, 0)$. The power of origin O w.r.t. this circle is $(\mu + R)(\mu - R) = \mu^2 - R^2 = 1$ so $R = \sqrt{\mu^2 - 1}$.
Then :

$$x = \mu + \sqrt{\mu^2 - 1}.\cos\alpha \qquad y = \sqrt{\mu^2 - 1}.\sin\alpha$$

We eliminate λ and find :

$$\mu = x - y/\tan\alpha \qquad \mu^2 = \frac{y^2 - \sin^2\alpha}{\sin^2\alpha}$$

$$(x - y/\tan\alpha)^2 = \frac{y^2 - \sin^2\alpha}{\sin^2\alpha}$$

$$x^2 - y^2 - \frac{2}{\tan\alpha}.x.y + 1 = 0$$

It is the same rectangular hyperbola as the one for Poncelet circles with $\alpha \to \pi/2 + \alpha$.

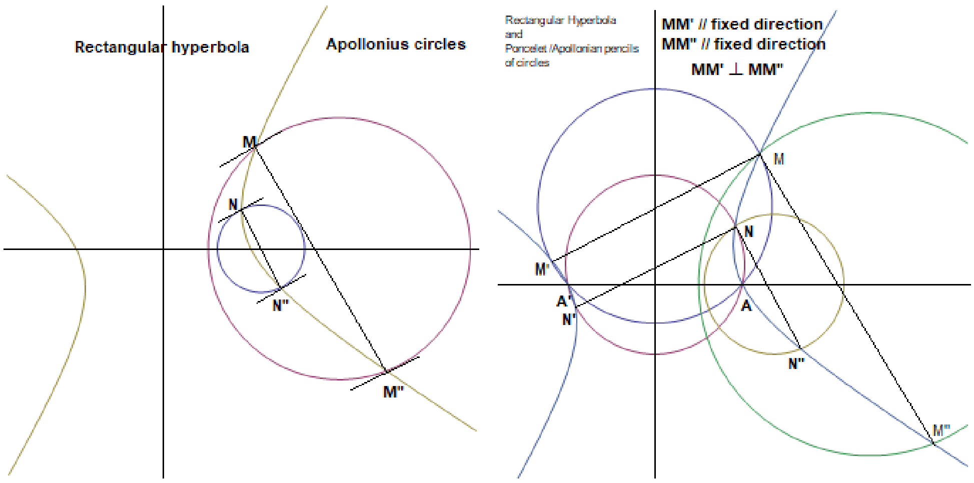

Figure 15: Pencils of circles and rectangular hyperbolas

If a secant parallel to a fixed direction cuts the hyperbola in M and M' then the middle I of MM' stays on a line through O the conjugate of the fixed direction. The point A' and A are the points of the hyperbola on line locus of I. The points of these circles of the pencil passing through A' and A where the tangent is parallel to a fixed direction is a rectangular hyperbola.
If a rectangular hyperbola and a circle have four common points two of which diametrally opposed on one of the curves, then the two others are diametrally opposed on the other.

15 The theorem of Brianchon-Poncelet

This theorem which goes back to 1820/21, is the following :
"A conic circonscribing a triangle ABC is a rectangular hyperbola if and only if it passes through the orthocenter and the center of this hyperbola lies on the nine point circle of the triangle". This can be proved by con-

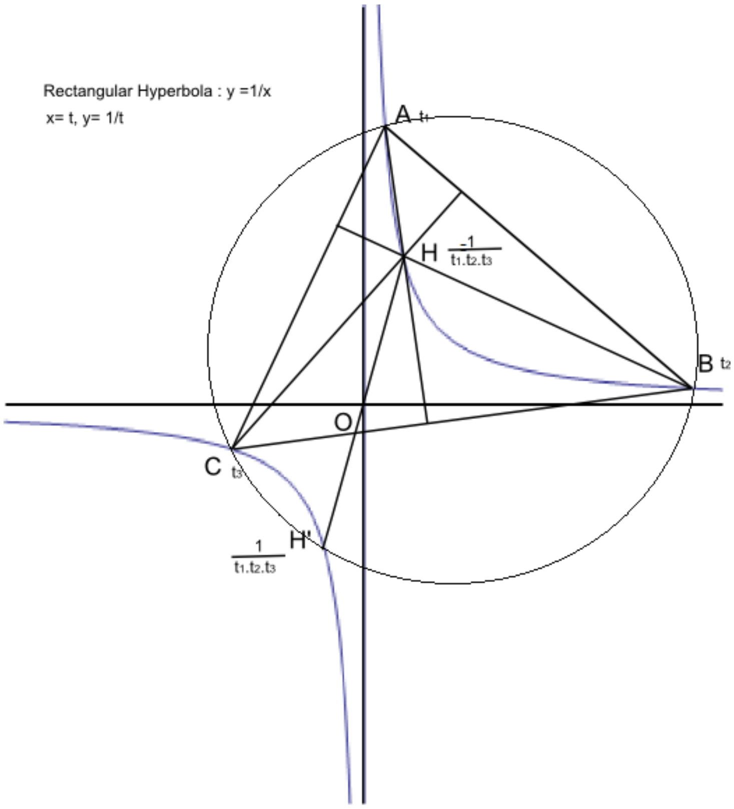

Figure 16: Orthocentric quadrilateral on Rectangular Hyperbola : xy=1

sidering the rectangular hyperbola with equation $x.y = 1$ in an orthogonal coordinates (see (5)). Coordinates of current point are $(t, 1/t)$, t the parameter.
We choose three point $A(t_1, 1/t_1), B(t_2, 1/t_2), C(t_3, 1/t_3)$. A few computations on orthogonal lines equations show that the perpendicular through A to BC cuts the hyperbola at a new point for $t = -1/(t_1.t_2.t_3)$ H which is the orthocentre of ABC. The symmetry of the formula confirms that the three heigths of the triangle pass through a common point on the hyperbola. The circumcenter of the triangle ABC meet again the hyperbola at H', the symmetric of H w.r.t. the center of the hyperbola. This completes the proof since cimcumscribed circle is the transformed of Euler circle of triangle ABC in a scaling with center at orthocenter H and ratio 2.

16 Power of a point w.r.t. a rectangular hyperbola

The well known power of a point w.r.t. a circle can be translated in a similar property of the rectangular hyperbola. For a circle this power is given by P(M) w.r.t. to circle (center O, radius R) by the constance of the product of line segments on the secants: $P(M) = MA.MB = MC.MD = d^2 - R^2$ where d is the distance MO from the center.

For the rectangular hyperbola there is an invariant product but on two

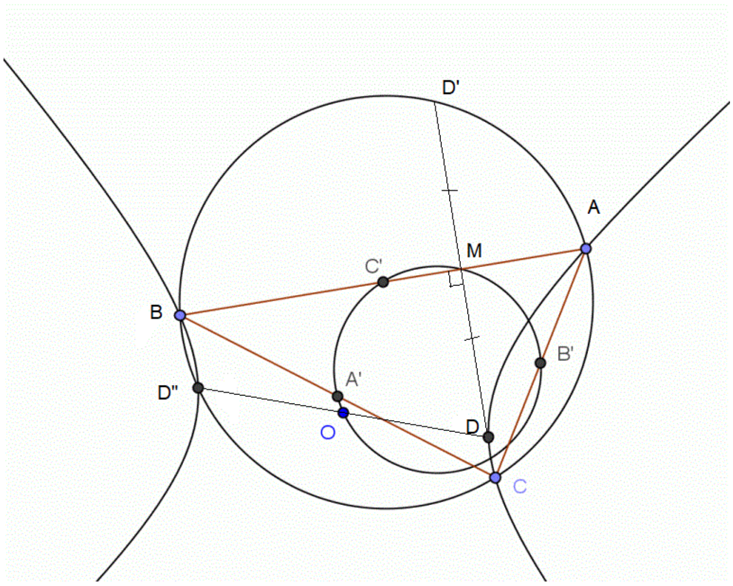

Figure 17: Orthocentric quadrilateral: MA.MB = – MC.MD

orthogonal lines. From a point M we can draw two such lines that cut the hyperbola in A and on one line and in C and D on the other. Then the product MA.MB=-MC.MD but it is not constant when the couple of orthogonal lines turn around M. The two orthogonal lines passing through M cut the hyperbola at four point A, B, C, D that form an orthocentric quadrilateral and a circle passing through 3, say A, B, and C, of the four points will pass through the symmetric of D w.r.t. the side AB of the triangle. So ABCD' are cocyclic and $\overline{MA}.\overline{MB} = \overline{MC}.\overline{MD'}$ and since $\overline{MD'} = -\overline{MD}$ so we have :

$$\overline{MA}.\overline{MB} = -\overline{MC}.\overline{MD}\text{" with "}AB \perp CD$$

17 Autopolar transformation of a logarithmic spiral w.r.t. a rectangular hyperbola

If a logarithmic spiral has the same center O as a rectangular hyperbola and if the two curves have a point of tangency then the logarithmic spiral is self transformed by the polarity w.r.t. the rectangular hyperbola (see fig.18). This property is a special example of autopolar plane curves (W-Kurven or anharmoniques) studied by Klein and Lie in 1871. See ref. (10) for details.

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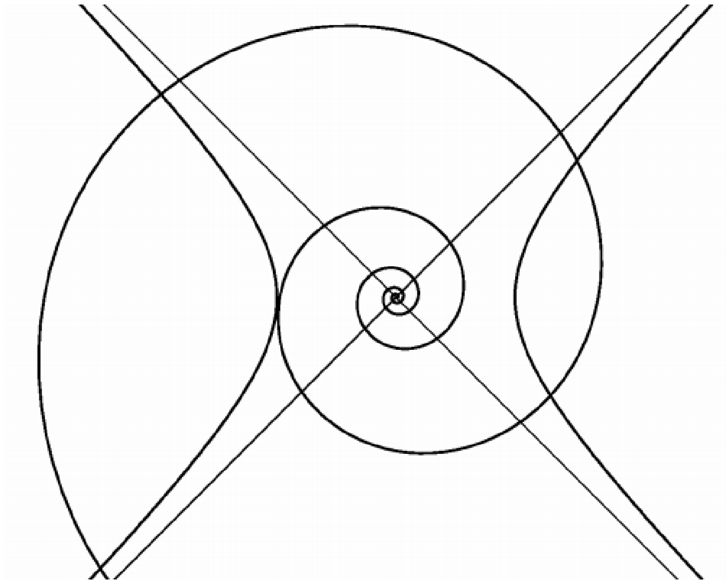

Figure 18: Autopolar logarithmic spiral w.r.t. rectangular hyperbola

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This article is the 23^{th} on plane curves.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.

Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643 - 1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.

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Gregory's transformation on the Web : http://christophe.masurel.free.fr